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ON THE VALUE MAXIMIZING PROPERTY OF INFINITE HORIZON EFFICIENT PROGRAMS*

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1. INTRODUCTION

An interesting problem in the theory of efficient allocation of resources over time, in an infinite horizon model, is to examine whether, in some appropriate sense, an efficient program maximizes the present-value of its consumption sequence.

A natural approach to determine the present-value of an efficient program is to evaluate the consumption sequence at the competitive (intertemporal profit maximizing) prices associated with it. But, here, we encounter a basic difficulty, since there might be technologies which generat efficient programs, whose associated competitive prices do not define a finite present value of consumption. (A technology which admits a "golden-rule" program is the best-known example.)

This brings us to the purpose of this note. It would be interesting to separate the technologies for which this difficulty *must* arise (for *some* efficient program generated by it) from those for which the difficulty cannot arise (for *any* efficient program generated by it). It is shown that every efficient program generated by the technology will have finite present value of consumption, at its competitive prices, iff the gross-output function, f, has one of the following characteristics: (i) f is strongly productive; (ii) f is strongly unproductive; (iii) f is a pure storage function at low input levels (Theorem 1).

It might be useful to interpret this result in terms of the usual neoclassical model of economic growth, where current output is given by a constant returns to scale production function, defined on capital and labor inputs, capital depreciates at a constant rate, and labor grows at a constant rate. In that framework, condition (i) [condition (ii)] means that the marginal product of capital is *uniformly* larger [smaller] than the sum of the depreciation rate and the labor growth rate. Condition (iii) means that for capital-labor ratios close to zero, the marginal product of capital is exactly equal to the depreciation rate plus the labor growth rate.

Using this result, it is shown that if the gross-output function, f, satisfies one of the above stated conditions, then efficiency of a feasible program is *equivalent* to the statement that it maximizes the present value of its consumption sequence, at its competitive prices, among all feasible programs (Theorem 2).

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2. THE MODEL

Consider an aggregative model, with a technology given by a gross-output function, f, from R_+ to itself. The technological possibilities are specified by inputs, x, and outputs $y=f(x)$, for $x \geq 0$.

The following assumptions on f are maintained throughout:

(A.1) $f(0)=0$;

(A.2) *f is strictly increasing for $x \geq 0$;*

(A.3) *f is concave for $x \geq 0$;*

(A.4) *f is differentiable for $x \geq 0$.*

A *feasible production program* from $\underline{x} \geq 0$, is a sequence $<x, y> = <x_t, y_{t+1}>$ satisfying

$$(1) \qquad x_0 = \underline{x},\ 0 \leq x_t \leq y_t \quad \text{for} \quad t \geq 1, \qquad f(x_t) = y_{t+1} \quad \text{for} \quad t \geq 0.$$

The *consumption program* $<c> = <c_t>$ generated by $<x, y>$ is given by

$$(2) \qquad c_t = y_t - x_t \ (\geq 0) \qquad \text{for} \quad t \geq 1.$$

The sequence $<x, y, c>$ is called a *feasible program* from $\underline{x}$, it being understood that $<x, y>$ is a production program, and $<c>$ is the corresponding consumption program.

A feasible program $<x', y', c'>$ from $\underline{x}$, *dominates* a feasible program $<x, y, c>$ from $\underline{x}$, if $c'_t \geq c_t$ for all $t \geq 1$, and $c'_t > c_t$ for some t. A feasible program $<x, y, c>$ from $\underline{x}$ is *inefficient* if there is a feasible program $<x', y', c'>$ from $\underline{x}$ which dominates it. A feasible program is called *efficient* if it is not inefficient.

The *competitive price sequence* $<p> = <p_t>$ associated with a feasible program $<x, y, c>$ is given by

$$(3) \qquad p_0 = 1, \quad p_{t+1} = p_t / f'(x_t) \qquad \text{for} \quad t \geq 0.$$

These are the prices which yield maximum *intertemporal profits*:

$$(4) \qquad w_t = p_{t+1} f(x_t) - p_t x_t \geq p_{t+1} f(x) - p_t x, \quad \text{for} \quad x \geq 0,\ t \geq 0.$$

The *input value sequence* $<v> = <v_t>$ associated with a feasible program $<x, y, c>$ is given by $v_t = p_t x_t$ for $t \geq 0$. A feasible program $<x, y, c>$ from $\underline{x}$ is said to have *bounded input value* if $\sup_{t \geq 0} v_t < \infty$. Otherwise, it is said to have *unbounded input value*. It is said to have *bounded consumption value* if the sequence $<p_t c_t>$ is summable; otherwise, it is said to have *unbounded consumption value*. A feasible program $<\bar{x}, \bar{y}, \bar{c}>$ from $\underline{x}$, is called the *pure accumulation program* from $\underline{x}$, if it satisfies $\bar{x}_{t+1} = f(\bar{x}_t)$ for $t \geq 0$.

The gross-output function, f, is said to be *productive* if $f'(x) > 1$ for $x \geq 0$; it is *strongly productive* if $\inf_{x \geq 0} f'(x) > 1$. Similarly, f is *unproductive* if $f'(x) < 1$ for $x \geq 0$; it is *strongly unproductive* if $\sup_{x \geq 0} f'(x) < 1$. Finally, f is a *pure storage function* if $f'(x) = 1$ for $x \geq 0$; it is a *pure storage function at low input levels*, if there is $x^* > 0$, such that $f'(x) = 1$ for $0 \leq x \leq x^*$.

3. A SUFFICIENT CONDITION FOR BOUNDED CONSUMPTION VALUE

It is well-known, in the literature, that there are technologies which can generate efficient programs which do not have bounded consumption value (for example, technologies which allow for the existence of a "golden-rule" program). It is equally well-known that there are special technologies which can only generate efficient programs with bounded consumption value (see, for example, McFadden [1967] and Benveniste [1976] for discussions of such technologies). However, there does not exist any simple characterization of technologies which distinguishes the above two types. This section, and the next, are devoted to obtaining such a characterization. To this end, consider the following condition on the gross-output function, f:

CONDITION B. f satisfies one of the following three properties: (i) *f is strongly productive*; (ii) *f is strongly unproductive*; (iii) *f is a pure storage function at low input levels.*

In this section, we will establish that Condition B is sufficient to guarantee that every efficient program has bounded consumption value.

We start by noting, without proofs, some well-known properties of concave functions, which will often be used later.

LEMMA 1. *Under* (A.1)–(A.4), *f satisfies the following properties*:

(5) *If* $x \geq x' > 0$, *then* $f'(x) \leq f'(x')$.

(6) *If* $x \geq x' > 0$, *then* $[f(x)/x] \leq [f(x')/x']$.

(7) *For* $x > 0$, $[f'(x)x/f(x)] \leq 1$.

(8) *If* $\sup_{x\geq 0} f'(x) < 1$, *then* $\sup_{x\geq 0} [f(x)/x] < 1$.

(9) *If* $\inf_{x\geq 0} f'(x) < 1$, *then* $\inf_{x\geq 0} [f(x)/x] < 1$.

(10) *If* $\sup_{x\geq 0} f'(x) < \infty$, *then* $\lim_{x\to 0} [f'(x)x/f(x)] = 1$.

(11) *If* $\inf_{x\geq 0} f'(x) > 0$, *then* $\lim_{x\to\infty} [f'(x)x/f(x)] = 1$.

We turn now to the main result of this section, which is given by

LEMMA 2. *Under* (A.1)–(A.4), *every efficient program* $<x, y, c>$ *from* $\underline{x} > 0$ *has bounded consumption value if Condition B is satisfied.*

PROOF. If Condition B (i) or (ii) is satisfied, it can be shown that the pure accumulation program from $\underline{x}$ has bounded input value. This, in turn, implies that every efficient program has bounded consumption value.

If Condition B (i) is satisfied, then the pure accumulation program $<\bar{x}, \bar{y}, \bar{c}>$

from $\underline{x}$ has bounded input value, by Corollary 1, p. 341, of Benveniste [1976]. If Condition B (ii) is satisfied, then since $\sup_{x\geq 0} f'(x)=e<1$, so by (10), $\sup_{x\geq 0}\cdot [f(x)/x]=e$. If $<\bar{x}, \bar{y}, \bar{c}>$ is the pure accumulation program from $\underline{x}$, then $\bar{x}_{t+1}=f(\bar{x}_t)\leq e\bar{x}_t$, and so $\bar{x}_t\to 0$ as $t\to\infty$. Next, note that $\bar{v}_{t+1}=[f(\bar{x}_t)/\bar{x}_t f'(\bar{x}_t)]\bar{v}_t$, so that by (7), $\bar{v}_t$ is monotonically non-decreasing.

Suppose, now, that $<\bar{x}, \bar{y}, \bar{c}>$ does not have bounded input value. Then, $\bar{v}_t\to\infty$, as $t\to\infty$. For $t\geq 0$, we have $(\bar{x}_{t+2}-\bar{x}_{t+1})=f(\bar{x}_{t+1})-f(\bar{x}_t)\geq f'(\bar{x}_{t+1})\cdot(\bar{x}_{t+1}-\bar{x}_t)$. Iterating on this relation, $(\bar{x}_{t+2}-\bar{x}_{t+1})\geq[\prod_{s=1}^{t+1}f'(\bar{x}_s)](\bar{x}_1-\bar{x}_0)$, so that we get $\bar{x}_{t+2}\geq\bar{x}_{t+1}-[\prod_{s=1}^{t+1}f'(\bar{x}_s)](\bar{x}_1-\bar{x}_0)$. Multiplying through by $\bar{p}_{t+2}$, we have $\bar{v}_{t+2}\geq\bar{p}_{t+2}\bar{x}_{t+1}-[(\bar{x}_0-\bar{x}_1)/f'(\bar{x}_0)]$. Dividing through by $\bar{v}_{t+1}$ and simplifying, we get $[f(\bar{x}_{t+1})/\bar{x}_{t+1}f'(\bar{x}_{t+1})]\geq[1/f'(\bar{x}_{t+1})]-[(\bar{x}_0-\bar{x}_1)/f'(\bar{x}_0)\bar{v}_{t+1}]$. Taking limits on both sides, we have $1\geq[1/f'(0)]$, which means $f'(0)\geq 1$, a contradiction.

Now, when Condition B (i) or (ii) is satisfied, it can be shown that every efficient program has bounded consumption value. Consider an efficient program $<x, y, c>$ from $\underline{x}>0$. Then, since $p_t\leq\bar{p}_t$, so $p_t c_t\leq\bar{p}_t c_t=\bar{p}_t(c_t-\bar{c}_t)$. Hence, for $T\geq 1$, we have, by using (4),

$$\sum_{t=1}^{T} p_t c_t \leq \sum_{t=1}^{T}\bar{p}_t(c_t-\bar{c}_t)\leq\bar{p}_T(\bar{x}_T-x_T)\leq\bar{p}_T\bar{x}_T. \tag{12}$$

Since we have shown that $\bar{p}_T\bar{x}_T$ is bounded above, so $<x, y, c>$ has bounded consumption value.

When, Condition B (iii) is satisfied, we must adopt a different proof. We will show, first, that for an efficient program $<x, y, c>$, x_t must converge to zero. Using this fact, we will then show that $<x, y, c>$ has bounded consumption value. We start with the first step. If $x_t=0$, for some finite t, then we are done. If not, then $x_t>0$ for $t\geq 0$. Then, $c_{t+1}=f(x_t)-x_{t+1}=[f(x_t)/x_t]x_t-x_{t+1}$. Since under Condition B (iii), $\sup_{x\geq 0}f'(x)=1$, so $\sup_{x\geq 0}[f(x)/x]=1$, by (5), (6) and (10). Hence, $c_{t+1}\leq x_t-x_{t+1}$, and so for $T\geq 0$, we have $\sum_{t=0}^{T}c_{t+1}\leq x_0-x_{T+1}\leq x_0$. Thus, $<c_t>$ is summable, and $\sum_{t=0}^{\infty}c_{t+1}\leq x_0$.

Now, suppose x_t does not converge to zero. Then, since $x_{t+1}\geq x_t$ for $t\geq 0$, so there is $b>0$, such that $x_t\geq b$ for $t\geq 0$, and hence $y_t\geq b$ for $t\geq 1$. This implies that the sequence $<c_t/y_t>$ is summable, so that by Lemma 2 in Mitra [1978], $<x, y, c>$ is inefficient, a contradiction. Hence, x_t must converge to zero, as $t\to\infty$.

By Condition B (iii), there is $x^*>0$, such that $f'(x)=1$ for $0\leq x\leq x^*$. Since x_t converges to zero, there is $\tau<\infty$, such that $x_t\leq x^*$ for $t\geq\tau$. By (7) and (10), $f(x)=x$ for $0\leq x\leq x^*$, so that $f(x_t)=x_t$ for $t\geq\tau$. Hence, for $T>\tau$, we have the following: $\sum_{t=\tau+1}^{T}p_t c_t=\sum_{t=\tau+1}^{T}[p_t f(x_{t-1})-p_t x_t]=\sum_{t=\tau+1}^{T}[p_{t-1}x_{t-1}-p_t x_t]\leq p_\tau x_\tau$. Hence, $\sum_{t=1}^{T}p_t c_t=\sum_{t=1}^{\tau}p_t c_t+\sum_{t=1\tau+1}^{T}p_t c_t\leq\sum_{t=1}^{\tau}p_t c_t+p_\tau x_\tau<\infty$.
This proves that $<x, y, c>$ has bounded consumption value.

REMARK. McFadden [1967] has shown that if $f(x)=dx$, where $d>0$, then every efficient program generated by this function has bounded consumption value. This can be seen to be a special case of Lemma 2 for if $f(x)=dx$, then f must satisfy Condition B (i) or (ii) or (iii), depending on whether $d>1$, or $d<1$, or $d=1$.

4. A NECESSARY CONDITION FOR BOUNDED CONSUMPTION VALUE

In this section, we will show that if Condition B is not satisfied then there is an efficient program which has unbounded consumption value. This means that Condition B is a necessary condition for every efficient program to have bounded value. The result will be obtained in two steps. First, it will be shown that if Condition B is violated, then the pure accumulation program from every positive initial input level has unbounded input value. Using this fact, we will then *construct* an efficient program with unbounded consumption value.

LEMMA 3. *Under* (A.1)–(A.4), *if Condition B is violated, then the pure accumulation program from every* $\underline{x}>0$, *has unbounded input value.*

PROOF. If Condition B is violated, then f must satisfy one of the following three conditions: (a) $\sup_{x\geq 0} f'(x)>1$ and $\inf_{x\geq 0} f'(x)<1$; (b) $\sup_{x\geq 0} f'(x)>1$ and $\inf_{x\geq 0} f'(x)=1$; (c) $\sup_{x\geq 0} f'(x)=1$ and $f'(x)<1$ for all $x>0$. We shall consider each of these cases in turn.

In case (a), by (7), $\sup_{x\geq 0} [f(x)/x]>1$, and by (8), $\inf_{x\geq 0} [f(x)/x]<1$. Note that $[f(x)/x]$ is continuous for $x>0$, so there is k, such that $f(k)=k$ and $0<k<\infty$. Also, there is k', such that $[f(k')/k']>1$, and $0<k'<k$. Let $\theta=(k'/k)$. Then, by (A.3), (A.4), we have $f(\theta k)-f(k)\leq f'(k)(\theta k-k)$. Hence, $f(k)-\theta k>f'(k)\cdot(k-\theta k)$. This means that $(k-\theta k)>f'(k)(k-\theta k)$, and $f'(k)<1$. It follows from this that k is the unique positive number for which $f(x)=x$. For, suppose there were another, call it $\hat{k}$. Then $f'(\hat{k})<1$, by the above arguments. Without loss of generality, then, we might suppose $k>\hat{k}$. Then, $(k-\hat{k})=f(k)-f(\hat{k})\leq f'(\hat{k})\cdot(k-\hat{k})<(k-\hat{k})$, a contradiction. Hence, $f(x)=x$ only for $x=k$. Thus, for $x<k$, we must have $x<f(x)<k$, and for $x>k$, we must have $x>f(x)>k$. This implies that if $<\bar{x}, \bar{y}, \bar{c}>$ is a pure accumulation program from $\underline{x}>0$, then $\bar{x}_t$ must converge to some value $\bar{k}>0$ as $t\to\infty$. By continuity of f, $f(\bar{k})=\bar{k}$, so that $\bar{k}=k$. Since $f'(k)<1$, so $\bar{p}_t\to\infty$, as $t\to\infty$. Hence, $<\bar{x}, \bar{y}, \bar{c}>$ has unbounded input value.

In case (b), we must have $f(x)>x$ for all $x>0$. This is because by (7), $f(x)\geq x$ for $x>0$, and if $f(\tilde{k})=\tilde{k}$ for some $\tilde{k}>0$, then by the arguments used in case (a), $f'(\tilde{k})<1$, which contradicts $\inf_{x\geq 0} f'(x)=1$. Now, consider the pure accumulation program $<\bar{x}, \bar{y}, \bar{c}>$ from $\underline{x}>0$. Since $f(x)>x$, for $x>0$, so $\bar{x}_t$ is monotonically increasing. If there is $K<\infty$, such that $\bar{x}_t\leq K$ for $t\geq 0$, then $\bar{x}_t$ converges to some $\bar{K}>0$. By continuity of f, $f(\bar{K})=\bar{K}$, a contradiction. Hence, $\bar{x}_t\to\infty$ as $t\to\infty$, and it follows that $f'(\bar{x}_t)\to 1$, as $t\to\infty$. Now, using the method used in the proof of Lemma 2 (in discussing Condition B (ii)), $[f(\bar{x}_{t+1})/\bar{x}_{t+1} f'(\bar{x}_{t+1})]\geq[1/f'(\bar{x}_{t+1})]-[(\bar{x}_0-\bar{x}_1)/f'(\bar{x}_0)\bar{v}_{t+1}]$ for $t\geq 0$. Suppose, now, that $<\bar{x}, \bar{y}, \bar{c}>$ has bounded input value, then since $\bar{v}_t$ is monotonically non-decreasing by (7), so it converges to some value $V<\infty$. Using this fact, and (11), and noting that $f'(\bar{x}_t)\to 1$ as $t\to\infty$, we have $1\geq 1+[(\bar{x}_1-\bar{x}_0)/f'(\bar{x}_0)V]$, by taking limits in the above inequality. Since $\bar{x}_1=f(\bar{x}_0)>\bar{x}_0$, we have a contradiction. Hence $<\bar{x}, \bar{y}, \bar{c}>$ has unbounded input value.

In case (c), we must have $f(x)<x$ for all $x>0$. To see this, note that, by (10), $\sup_{x\geq 0}[f(x)/x]=1$. Hence, $f(x)\leq x$ for $x>0$. Suppose, for some $\tilde{x}>0$, we have $f(\tilde{x})=\tilde{x}$, then by (6), $f(x)=x$ for all x satisfying $0<x\leq\tilde{x}$. But this means that for $x=\frac{1}{2}\tilde{x}$, $f'(x)=1$, a contradiction. Consider now the pure accumulation program $<\bar{x}, \bar{y}, \bar{c}>$ from $\underline{x}>0$. Since $f(x)<x$, for $x>0$, so $\bar{x}_t$ is monotonically decreasing, and converges to some value $\underline{k}$. If $\underline{k}>0$, then by continuity of f, $f(\underline{k})=\underline{k}$, a contradiction. Hence $\underline{k}=0$, and $\bar{x}_t\to 0$ as $t\to\infty$, so that $f'(\bar{x}_t)\to 1$ as $t\to\infty$. Now, note that for $t\geq 0$, $(\bar{x}_{t+2}-\bar{x}_{t+1})=f(\bar{x}_{t+1})-f(\bar{x}_t)\leq f'(\bar{x}_t)(\bar{x}_{t+1}-\bar{x}_t)$. Iterating on this relation, and simplifying, $\bar{x}_{t+2}\leq\bar{x}_{t+1}+[\prod_{s=0}^{t}f'(\bar{x}_s)](\bar{x}_1-\bar{x}_0)$. Multiplying through by $\bar{p}_{t+2}$, $\bar{v}_{t+2}\leq[\bar{v}_{t+1}/f'(\bar{x}_{t+1})]-[(\bar{x}_0-\bar{x}_1)/f'(\bar{x}_{t+1})]$. Suppose that $<\bar{x}, \bar{y}, \bar{c}>$ has bounded input value. Then by (7), since $\bar{v}_t$ is monotonically non-decreasing, so $\bar{v}_t$ converges to some value $\bar{V}<\infty$. Then, taking limits in the above inequality, $\bar{V}\leq\bar{V}-(\bar{x}_0-\bar{x}_1)$, a contradiction, since $\bar{x}_1=f(\bar{x}_0)<\bar{x}_0$. Hence, $<\bar{x}, \bar{y}, \bar{c}>$ must have unbounded input value.

LEMMA 4. *Under* (A.1)–(A.4), *if the pure accumulation program from every* $\underset{\sim}{x}>0$, *has unbounded input value, then there exists an efficient program with unbounded consumption value.*

PROOF. We will construct the required efficient program $<x, y, c>$. Given any $\underline{x}>0$, let $x_0=\underline{x}$, and $x_{t+1}=f(x_t)$ for $t=0, 1,\ldots, t_1$, where t_1 is the smallest integer, for which $p_t x_t\geq 1$. Since the pure accumulation program from $\underline{x}>0$, has unbounded input value, $t_1<\infty$. Let $y_{t+1}=f(x_t)$, $x_{t+1}=(1/2p_{t+1})$, $c_{t+1}=y_{t+1}-x_{t+1}$, for $t=t_1$. Then, it is easy to check that $p_{t+1}x_{t+1}=\frac{1}{2}$, $p_{t+1}c_{t+1}\geq\frac{1}{2}$ for $t=t_1$.

The rest of the program is defined in the following way. For $s\geq 1$, let $x_{t+1}=f(x_t)$, for $t=t_s+1,\ldots, t_{s+1}$, where t_{s+1} is the smallest integer, such that $p_{t_{s+1}}x_{t_{s+1}}\geq 1$. Define $y_{t+1}=f(x_t)$, $x_{t+1}=[1/(s+2)p_{t+1}]$ and $c_{t+1}=y_{t+1}-x_{t+1}$ for $t=t_{s+1}$. Then, it is easy to check that $p_{t+1}x_{t+1}=[1/(s+2)]$, and $p_{t+1}c_{t+1}\geq\frac{1}{2}$, for $t=t_{s+1}$. The proof that this sequence is a feasible program is completed by induction. Suppose the sequence has been defined up to $t_{s^*}+1$. Then, $x_{t_{s^*}+1}>0$, and so, since the pure accumulation program from $x_{t_{s^*}+1}$ has unbounded input value, so we can find $t_{s^*+1}<\infty$, such that $p_t x_t\geq 1$, for $t=t_{s^*+1}$. Noting that the sequence was well defined for $s^*=1$, we conclude that $<x, y, c>$ is a feasible program.

Notice that since $p_t>0$, for $t\geq 0$, and $p_t x_t=[1/(s+1)]$ for $t=t_s+1$, so $<x, y, c>$ is efficient, by Malinvaud [1953, Lemma 5]. Also, $p_t c_t\geq\frac{1}{2}$ for $t=t_s+1$, and so $<x, y, c>$ has unbounded consumption value.

We can now combine the results of Lemmas 2, 3, 4, to obtain the following complete characterization of technologies for which every efficient program generated by it has bounded consumption value.

THEOREM 1. *Under* (A.1)–(A.4), *every efficient program from* $\underline{x}>0$, *has bounded consumption value if and only if Condition B is satisfied.*

5. CONSUMPTION VALUE MAXIMIZATION

Theorem 1 tells us that if Condition B is not satisfied, then there will be some efficient program for which consumption value maximization, at its competitive prices, in the class of all feasible programs, will not hold (since such maximization will not make sense when consumption value is unbounded). On the other hand, if Condition B is satisfied, then since the consumption value of every efficient program must be bounded, we expect that every efficient program will also maximize consumption value, at its competitive prices, among all feasible programs. That this is indeed true is demonstrated in this section, in Theorem 2.

THEOREM 2. *Under* (A.1)–(A.4), *if Condition B is satisfied, then the following four statements are equivalent*:

(a) $<x, y, c>$ *is efficient from* $\underline{x}>0$.

(b) $\sum_{t=1}^{\infty} p_t c_t \geq \sum_{t=1}^{\infty} p_t c_t'$ *for every feasible program* $<x', y', c'>$ *from* $\underline{x}$.

(c) $\lim_{t\to\infty} p_t x_t = 0$.

(d) $p_0 x_0 + \sum_{t=0}^{\infty} w_t = \sum_{t=1}^{\infty} p_t c_t$.

PROOF. We will show that (a) implies (b) which implies (c) which implies (a). Also, that (c) implies (d), and (d) implies (c).

If (a) holds, then by Theorem 1, $<x, y, c>$ has bounded consumption value. Hence, by the corollary to the theorem in Cass and Yaari [1971, p. 338], (b) holds.

If (b) holds, and (c) is violated, then there is a subsequence of periods, and $m>0$, such that $v_t \geq m$ for this subsequence. Choose T such that $\sum_{t=T}^{\infty} p_t c_t \leq \frac{1}{2} m$. Choose $\tau \geq T$, such that $v_\tau \geq m$, and define a sequence $<x', y', c'>$ as follows: $(x_t', y_t', c_t') = (x_t, y_t, c_t)$ for $1 \leq t \leq \tau-1$, $x_0' = \underline{x}$, $(x_t', y_t', c_t') = (0, y_t, y_t)$ for $t=\tau$, and $(x_t', y_t', c_t') = (0, 0, 0)$ for $t>\tau$. Then, clearly, $<x', y', c'>$ is a feasible program from $\underline{x}$, and $\sum_{t=1}^{\infty} p_t c_t' \geq \sum_{t=1}^{\tau} p_t c_t + m \geq \sum_{t=1}^{\infty} p_t c_t + \frac{1}{2} m$, which contradicts (b).

If (c) holds, then either $x_t>0$ for $t \geq 0$, or there is $t_1<\infty$, such that $x_{t_1}=0$. In the former case, $p_t>0$, for $t \geq 0$, so by Malinvaud [1953, Lemma 5], $<x, y, c>$ is efficient. In the latter case, by Cass [1972, p. 203–4], $<x, y, c>$ is efficient. In either case, (a) holds.

If (c) holds, then $<x, y, c>$ is efficient, as noted above, so that by Theorem 1, it has bounded consumption value. Now, for $T \geq 1$,

$$\sum_{t=1}^{T} p_t c_t = \sum_{t=1}^{T} (p_t y_t - p_t x_t) = p_0 \underline{x} + \sum_{t=0}^{T-1} w_t - p_T x_T. \tag{13}$$

Since v_t has a limit, and so has $\sum_{t=1}^{T} p_t c_t$, so $\sum_{t=0}^{T-1} w_t$ has a limit. Taking limits in (13), and using (c) yields (d).

If (d) holds, we know that all terms in (13) have limits. Taking limits in (13), and using (d) yields (c).

REMARKS. (i) The statement that (a)⇔(b) says that efficiency is equivalent to maximization of the value of consumption, at its competitive price sequence, in the set of all feasible programs.

(ii) The statement that (a)⇔(c) says that efficiency is equivalent to no overaccumulation of input, where this overaccumulation is signalled by the transversality condition (c) being violated.

(iii) The statement that (a)⇔(d) says that efficiency is equivalent to the full utilization of "wealth" for consumption.

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